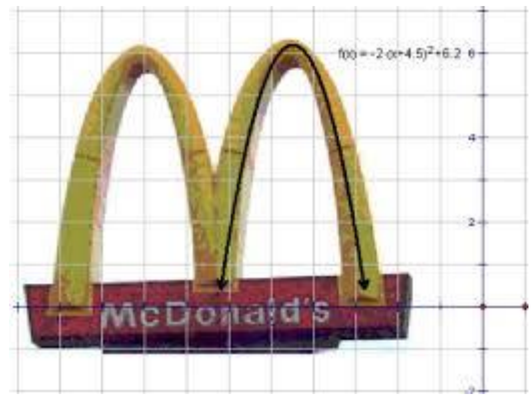
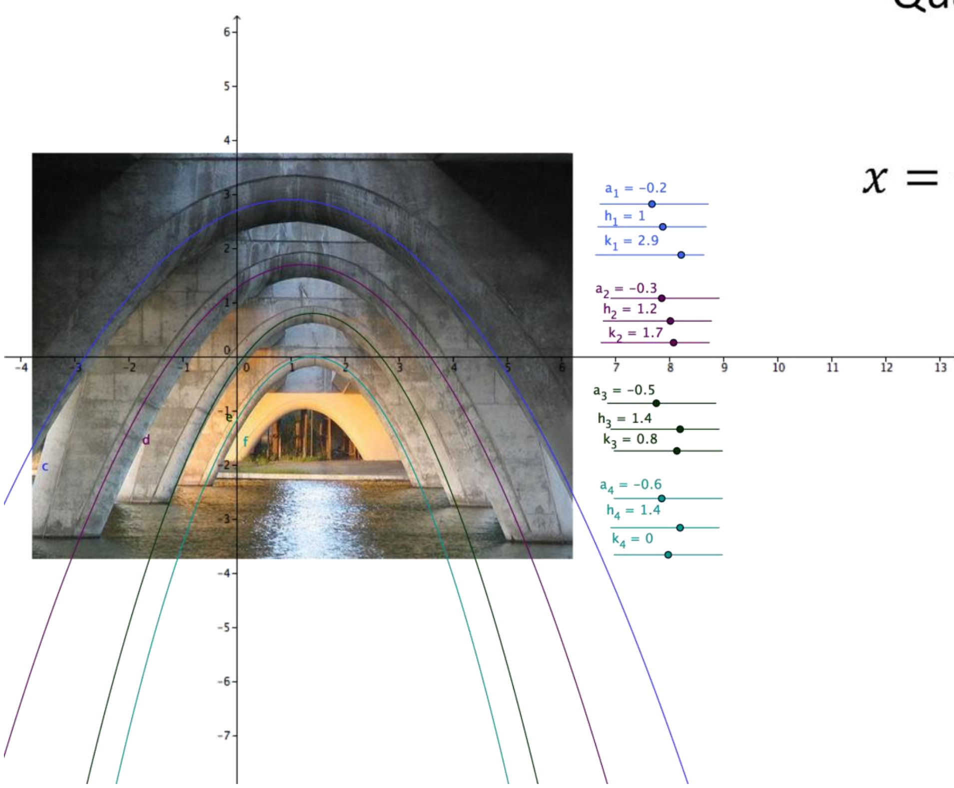
Unit 6





Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

6.1 Factoring Polynomials

Factoring is the inverse operation of multiplying polynomials. Factoring writes a polynomial as a product of its factors. The type of factoring used depends on the number and types of terms in the polynomial.

Factoring Common factors

To factor common factors, we use the distributive property in the form ab + ac =a (b+c). We first identify the common factor of all terms in the polynomial. Then, we determine the polynomial that occurs when the common factor is divided out of the polynomial.

Examples:

1. Factor: $15x^2 + 40x$

   Solution:
   $15x^2 + 40x$ has a common factor of 5x since 5x is a factor of both terms.
   $15x^2 + 40x = 5x\,(3x + 8)$

2. Factor: $4x^3 – 8x^2$

   Solution:
   $4x^3 – 8x^2$ has a common factor of $4x^2$ since $4x^2$ is a factor of both terms.
   $4x^3 – 8x^2 = 4x^2(x – 2)$

3. Factor: $6x^3y^2 + 9x^2y – 12xy$

   Solution:
   $6x^3y^2 + 9x^2y – 12xy$ has a common factor of 3xy.
   $6x^3y^2 + 9x^2y – 12xy = 3xy(2x^2y + 3x – 4)$

Factoring a Difference of Two Squares

A difference of two squares will be a polynomial with two terms where both terms are perfect squares and the terms are subtracted. A difference of two squares will look like $a^2 – b^2$ and will factor into $(a + b)(a – b)$.

Examples:

1. Factor: $4x^2 – 25$

   Solution:
   $4x^2 – 25 = (2x + 5)(2x – 5)$

2. Factor: $9x^2 - 16$

   Solution:
   $9x^2 - 16 = (3x + 4)(3x - 4)$

Factoring a Sum of Difference of Two Cubes
A sum or difference of cubes will be a binomial in which both terms are perfect cubes. The formulas to factor a sum or difference of cubes are:

1. $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
2. $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Examples:

1. Factor: $8x^3 + 27$

   Solution:

$8x^3 + 27$	First rewrite in factored form to identify a and b
$= (2x)^3 + 3^3$	Use the formula for sum of two cubes
$= (2x + 3)((2x)^2 - (2x)(3) + 3^2)$	Simplify
$= (2x + 3)(4x^2 - 6x + 9)$	

2. Factor: $x^3 - 125y^3$

   Solution:

$x^3 - 125y^3$	Rewrite to identify a and b
$= (x)^3 - (5y)^3$	Use the formula for the difference of two cubes
$= (x - 5y)(x^2 + (x)(5y) + (5y)^2)$	Simplify
$= (x - 5y)(x^2 + 5xy + 25y^2)$	

Factoring by Grouping
When a polynomial has four terms, after looking for common factors, try factoring by grouping. Just like it says, factoring by grouping means that you will group terms with common factors before factoring. This is done by grouping two pairs of terms. Then, factoring each pair of terms.

Examples:

1. Factor: $x^3 - 2x^2 + 6x - 12$

   Solution:

$x^3 - 2x^2 + 6x - 12$	Group into two pairs of terms
$= (x^3 - 2x^2) + (6x - 12)$	Factor out the common factor from both pairs
$= x^2(x - 2) + 6(x - 2)$	Factor out the common factor of (x – 2)
$= (x - 2)(x^2 + 6)$	

2. Factor: $3x^3 + 12x^2 - 5x - 20$

Solution:

$3x^3 + 12x^2 - 5x - 20$	Group into two pairs of terms
$= (3x^3 + 12x^2) + (-5x - 20)$	Factor out the common factor from both pairs
$= 3x^2(x + 4) - 5(x + 4)$	Factor out the common factor of (x + 4)
$= (x + 4)(3x^2 - 5)$	

Factoring by grouping can also be used to factor trinomials. The polynomial must first be rewritten with 4 terms where the middle term is split into two terms.

Steps: Given $ax^2 + bx + c$:

1. Multiply a and c.
2. Find factors of ac which add to b.
3. Split the term bx into two parts using the factors found.
4. Group the 4 terms into two pairs.
5. Factor by grouping.

Example:

Factor: $11x^2 - 41x - 12$

Solution:

Multiply 11 and -12 which is -132.
Then, find factors of -132 that will add up to -41.
The factors are -44 and 3.
Split the middle term: $11x^2 - 41x - 12 = 11x^2 - 44x + 3x - 12$
Factor by grouping:
$11x^2 - 44x + 3x - 12 = 11x(x - 4) + 3(x - 4) = (x - 4)(11x + 3)$

Factoring Trinomials of the form $x^2 + bx + c$

When factoring trinomials of this form, the factors are two binomials which each have a first term of x. The second terms must multiply to be c and add to be b.
To find these numbers, keep the following in mind.

- If c is positive, then the factors you're looking for are either both positive or else both negative.
  If b is positive, then the factors are positive
  If b is negative, then the factors are negative.
  In either case, you're looking for factors that add to b.

- If c is negative, then the factors you're looking for are of alternating signs;
  that is, one is negative and one is positive.
  If b is positive, then the larger factor is positive.
  If b is negative, then the larger factor is negative.
  In either case, you're looking for factors that are b units apart.

Examples:

1. Factor: $x^2 + 10x + 21$

   Solution: We are looking for two numbers which multiply to 21 and add to 10.
   $x^2 + 10x + 21 = (x + 7)(x + 3)$

2. Factor: $x^2 - 15x + 36$

   Solution: We are looking for two numbers which multiply to 36 and add to -15.
   $x^2 - 15x + 36 = (x - 12)(x - 3)$

3. Factor: $x^2 + 3x - 28$

   Solution: We are looking for two numbers which multiply to -28 and add to 3.
   $x^2 + 3x - 28 = (x + 7)(x - 4)$

Factoring Trinomials of the form $ax^2 + bx + c$

When factoring trinomials of this form, the factors are two binomials which have first terms which multiply to ax. The second terms must multiply to be c. The product of the outside terms plus the product of the inside terms must add to bx.
To find these numbers, keep the following in mind.

- If the polynomial does not have a common factor, then neither of the factors will have a common factor.
- If c is positive, then the factors you're looking for are either both positive or else both negative.
  If b is positive, then the factors are positive.
  If b is negative, then the factors are negative.
- If c is negative, then the factors you're looking for are of alternating signs;
  that is, one is negative and one is positive.

Examples:

1. Factor: $4x^2 + 49x + 55$

   Solution: The first terms must multiply to $4x^2$ so they will be either 2x and 2x or x and 4x. The last terms multiply to 55 so they must be 1 and 55 or 5 and 11. The product of the outside terms and inside terms must add to 49x.
   $4x^2 + 49x + 55 = (x + 11)(4x + 5)$

2. Factor: $2x^2 - 3x - 5$

   Solution: The first terms must multiply to $2x^2$ so they will be 2x and x. The last terms must multiply to -5 so they must be opposite signs. The outside and inside products must add to -3x.
   $2x^2 - 3x - 5 = (2x - 5)(x + 1)$

3. Factor: $8x^2 – 2x – 3$

Solution: The first terms must multiply to $8x^2$ so they will either be 8x and x or 4x and 2x. The last terms must multiply to -3.
(4x – 3)(2x + 1)

Notes:

1. Always look for common factors first.

2. If the polynomial has two terms, look for a difference of squares or a sum or difference of cubes.

3. If a polynomial has 4 terms, try factoring by grouping.

4. If a polynomial has 3 terms, then use trial and error or factoring by grouping or other method.

Homework:
Factor the following, if possible:

1. $7x^2 – 21x$
2. $6x^3 – 12x^2 + 15x$
3. $8x^3 + 4x^2$
4. $t^2 – 81$
5. $y^3 + 27$
6. $64h^3 – 1$
7. $3x^2 – 27$
8. $4d^2 – 25$
9. $x^2 + 6x + 8$
10. $x^2 + 3x$
11. $x^2 + x − 6$
12. $x^2 − x − 12$
13. $x^2 + 6x + 8$
14. $x^2 + 15x + 56$
15. $x^2 − 4x + 3$
16. $x^2 − 2x – 3$
17. $x^2 – 16$
18. $x^2 + 18x + 80$
19. $x^2 − 7x + 6$
20. $x^2 + 17x + 72$
21. $x^2 − 9x$
22. $x^2 + 8x + 12$
23. $− 35x + 63$
24. $− 4x^2 + 43x – 63$
25. $− 60x^2 + 34x – 4$
26. $4x^2 − 15x + 14$
27. $3x^2 + 7x + 2$
28. $x^2 - 49$
29. $x^2 + 49$
30. $3x^4 + 9x^2$

31. $x^2 + 2x – 6$

32. $x^2 - 21x – 72$

33. $4x^2 + 8x + 4$

34. $7x^2 + 5x - 56$

35. $x^4 - 10x^2 + 9$

36. $33x^2 - x – 14$

37. $4x^3 + 5x^2 + 28x + 35$

38. $x^3 + 2x^2 + 7x + 14$

39. $3x^3 – 6x^2 + 7x – 14$

40. $6x^3 – 6x^2 – 10x + 10$

41. $10x^2 + 35x + 15$

42. $6x^2 + x – 1$

43. $4t^2 – 5t – 6$

44. $3y^2 + 13y + 4$

45. $8m^2 – 10m – 3$

46. $8k^2 + 2k – 15$

47. $3a^2 + 10a + 7$

48. $2a^2 – 17a + 30$

49. $8x^2 – 14x + 3$

50. $14r^2 + 16r + 2$

51. $5a^2 – 6 – 7a$

52. $15p^2 – p – 6$

53. $4r^2 + r – 3$

54. $11s + 12s^2 – 5$

55. $6q^2 + 23q + 21$

56. $9p^2 + 6p –24$

57. $3r^2 + r – 10$

58. $6x^2 – x – 12$

59. $125x^3 + 8y^3$

60. $5n^3 – 40$

6.2 Solving Quadratric Equations

A quadratic equation is any equation which can be written in the form $ax^2 + bx + c = 0$. There are several ways to solve quadratic equations including graphically, numerically, and several algebraic methods. Quadratic equations can have zero, one or two real solutions. If the quadratic equation has zero real solutions, then it will have two complex solutions.

When solving an equation of the type $x^2 = k$, we will get two real solutions of the form $x = \pm\sqrt{k}$. You can see this when solving $x^2 = 25$. Both 5 and -5 when squared will give a result of 25.

Solving by taking square roots

In order to solve by taking square roots, the term that is squared must be the only term with a variable and must be able to be isolated on one side of the equal sign.

Steps:

1. Isolate the term that is squared on one side of the equal sign.
2. Take the square root of both sides of the equation. Make sure to put a plus or minus.
3. If needed, separate into two equations and solve each.
4. Verify the solutions.

Examples:

1. Solve $5x^2 – 7 = 13$ for x.

   Solution:

$5x^2 – 7 = 13$	Add 7 to both sides
$5x^2 = 20$	Divide both sides by 5
$x^2 = 4$	Take the square root of both sides (plus or minus)
$\sqrt{x^2} = \pm\sqrt{4}$	Simplify
$x = \pm 2$	

2. Solve $2(3x + 1)^2 = 18$ for x.

   Solution:

$2(3x + 1)^2 = 18$	Divide both sides by 2
$(3x + 1)^2 = 9$	Take the square root of both sides
$\sqrt{(3x+1)^2} = \pm\sqrt{9}$	Simplify
$3x + 1 = \pm 3$	Split into two equations to solve
$3x + 1 = 3$ or $3x + 1 = -3$	Solve each equation for x
$3x = 2$ or $3x = -4$	
$x = \frac{2}{3}$ or $x = -\frac{4}{3}$	

Solving by Factoring

To solve by factoring, the polynomial must be factorable and then we use the zero-factor principle.

Zero-Factor Principle: If $a \cdot b = 0$ then $a = 0$ or $b = 0$ or both are equal to zero.

If a polynomial can be factored then this method can be used to find the solutions to the quadratic equation. If the polynomial is not factorable, then you will need to use another method.

Steps to solve by factoring:

1. Put the polynomial in standard form (set it equal to zero).
2. Factor the polynomial.
3. Set each factor equal to zero.
4. Solve each equation.

Examples:

1. Solve $x^2 + 16x + 48 = 0$ by factoring.

   Solution:

$x^2 + 16x + 48 = 0$	Factor the quadratic
$(x + 12)(x + 4) = 0$	Use the zero-factor principle
$x + 12 = 0$ or $x + 4 = 0$	Solve each equation
$x = -12$ or $x = -4$	

2. Solve $3x^2 - x = 10$ by factoring.

   Solution:

$3x^2 - x = 10$	Set the equation equal to zero
$3x^2 - x - 10 = 0$	Factor the polynomial
$(3x + 5)(x - 2) = 0$	Set each factor equal to zero
$3x + 5 = 0$ or $x - 2 = 0$	Solve each equation
$x = -5/3$ or $x = 2$	

Solving by Completing the Square

The method of completing the square involves rewriting the equation so that the equation can be solved by taking square roots. The equation is first rewritten to contain a perfect square trinomial. Recall that a perfect square trinomial is a trinomial of the form where $x^2 + bx + c = (x + e)^2$ such as $x^2 + 10x + 25 = (x + 5)^2$.

Steps to complete the square:

1. Rewrite the equation so that the terms with variables are on one side of the equation and constants are on the other side of the equation.
2. If the leading coefficient is not 1, divide both sides by an appropriate constant so that the leading coefficient is 1.

3. Complete the square by adding an appropriate constant to the side with the variables so that you have a perfect square trinomial. This is done by taking half of the coefficient of x and squaring it.
4. Factor the perfect square trinomial.
5. Solve the equation by taking square roots.

Examples:

1. Solve $x^2 - 6x - 4 = 0$ by completing the square.

Solution:

$x^2 - 6x - 4 = 0$	
$x^2 - 6x = 4$	First rewrite the equation.
$x^2 - 6x + 9 = 4 + 9$	Since the leading coefficient is 1, complete the square by taking half of the coefficient of x and squaring. $(-6/2)^2 = 9$ Add this to both sides of the equation.
$(x - 3)^2 = 13$	Factor the perfect square trinomial.
$x - 3 = \pm\sqrt{13}$	Take the square root of both sides of the equation.
$x = 3 \pm\sqrt{13}$	Solve for x.

2. Solve $2x^2 + 8x - 17 = 0$ by completing the square.

Solution:

$2x^2 + 8x = 17$	Rewrite the equation.
$x^2 + 4x = 17/2$	Since the leading coefficient is not 1, divide both sides of the equation by the leading coefficient of 2.
$x^2 + 4x + 4 = 17/2 + 4$	Complete the square by taking $(4/2)^2 = 4$ and adding this number to both sides of the equation.
$(x + 2)^2 = 25/2$	Factor the perfect square trinomial.
$x + 2 = \pm\sqrt{\frac{25}{2}}$	Take the square root of both sides of the equation.
$x = -2 \pm\sqrt{\frac{25}{2}} = -2 \pm\frac{\sqrt{10}}{2}$	Solve for x. Rationalize the denominator for a "simplified" solution.

3. Solve $3x^2 + 24x + 75 = 0$ by completing the square.

Solution:

$3x^2 + 24x = -75$	Rewrite the equation.
$x^2 + 8x = -25$	Divide through by 3 to have a leading coefficient of 1.
$x^2 + 8x + 16 = -25 + 16$	Complete the square by taking $(8/2)^2 = 16$ and adding this number to both sides of the equation.
$(x + 4)^2 = -9$	Factor the perfect square trinomial.
$x + 4 = \pm\sqrt{-9}$	Take the square root of both sides of the equation. Recall that the square root of -1 is the imaginary number *i*.
$x = -4 \pm 3i$	Solve for x.

Solving using the Quadratic Formula

The quadratic formula will solve any quadratic equation. The equation must first be put in standard form $ax^2 + bx + c = 0$.

Quadratic Formula: The solutions to the equation $ax^2 + bx + c = 0$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Derivation of the quadratic formula: The quadratic equation can be derived by solving the general quadratic $ax^2 + bx + c = 0$ for x using the method of completing the square.

$ax^2 + bx = -c$	Move the constant.
$x^2 + \frac{b}{a}x = -\frac{c}{a}$	Divide both sides of the equation by the leading coefficient.
$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$	Complete the square. $\left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$
$(x + \frac{b}{2a})^2 = \frac{b^2-4ac}{4a^2}$	Factor the perfect square trinomial on the left and add the terms on the right side of the equation by getting a common denominator.
$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2-4ac}}{2a}$	Take the square root of both sides of the equation.
$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$	Solve for x.

Steps to use the quadratic formula:

1. Put the equation in standard form, or in other words, set the equation equal to zero.
2. Identify the values of a, b, and c.
3. Plug the values of a, b, and c into the quadratic formula.
4. Simplify.
5. Approximate the solutions if necessary.

Examples:

1. Solve $20x^2 + 13x + 2 = 0$ using the quadratic formula.

   Solution:
   From the equation we can see that a = 20, b = 13, and c = 2.
   Plugging into the quadratic equation, we get:

   $x = \frac{-13\pm\sqrt{13^2-4(20)(2)}}{2(20)}$ Simplifying

   $x = \frac{-13\pm\sqrt{169-160}}{40}$

   $x = \frac{-13\pm\sqrt{9}}{40}$ Taking the square root

$$x = \frac{-13 \pm 3}{40}$$ Splitting into two equations

$$x = \frac{-13+3}{40} = \frac{-10}{40} = \frac{-1}{4} \quad \text{or} \quad x = \frac{-13-3}{40} = \frac{-16}{40} = \frac{-2}{5}$$

The solutions are x =-1/4 and x = -2/5.

2. Solve $2x^2 - 3x + 7 = 0$ using the quadratic formula.

Solution:
From the equation we can see that a = 2, b = -3, and c = 7.
Plugging into the quadratic equation, we get:

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(7)}}{2(2)}$$ Simplifying

$$x = \frac{3 \pm \sqrt{9-56}}{4}$$

$$x = \frac{3 \pm \sqrt{-47}}{4}$$ Simplifying the square root of -1.

$$x = \frac{3 \pm \sqrt{47}i}{4}$$ The solutions are x = $\frac{3+\sqrt{47}i}{4}$ and x = $\frac{3-\sqrt{47}i}{4}$.

Solving Numerically

To solve a quadratic equation numerically, we use a table. Remember that most quadratic equations have two solutions. To solve $ax^2 + bx + c = 0$ numerically, we let $y_1 = ax^2 + bx + c$ and look for the values of x which have an output value of zero.

Example:

Solve $2x^2 - 5x + 2 = 0$ numerically.

Solution:

Set $y_1 = 2x^2 - 5x + 2$ and make a table. Find the values where $y_1 = 0$.

X	Y_1	
0	2	
.5	0	
1	-1	
1.5	-1	
2	0	
2.5	2	
3	5	

Press + for ΔTbl

The solutions are x = 0.5 and x = 2.

Solving quadratic equations numerically is often difficult because the solutions are not usually integer values. A better way to use technology to solve quadratic equations is to solve them graphically.

Solving Graphically

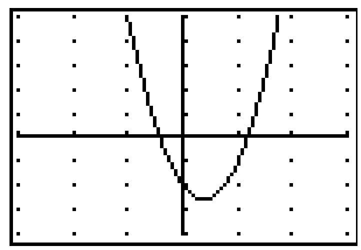
When solving $ax^2 + bx + c = 0$ graphically, we first set $y_1 = ax^2 + bx + c$. Notice that the solutions will be at the values for which $y_1 = 0$. Points which have a y-value of zero are the x-intercepts of a graph. So, to find the solutions we simply need to find the x-intercepts of the graph. This can be done on the calculator in two ways. The first is to set $y_2 = 0$ and find the intersection of the two graphs. The second way is to use the **ZERO** key which is located in the **CALC** menu (**2nd TRACE**). If the graph is already given, then estimate the x-intercepts from the graph.

Examples:

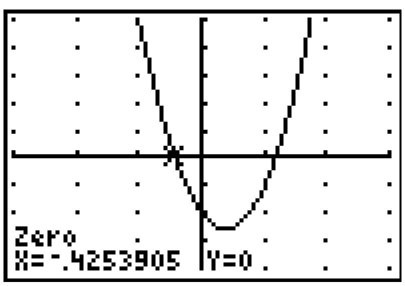
1. Solve $4x^2 - 3x - 2 = 0$ graphically.

    Solution:

    Set $y_1 = 4x^2 - 3x - 2$ and find an appropriate viewing window to see the x-intercepts.

    Use the **ZERO** command twice to find the two x-intercepts.

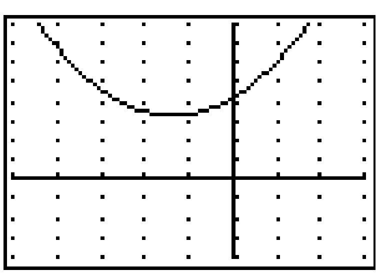
    

    So, the solutions are x = -0.4254 and x = 1.1754.

2. Solve $0.5x^2 + 1.35x + 4.2 = 0$ graphically.

    Solution:

    Set $y_1 = 0.5x^2 + 1.35x + 4.2$ and find an appropriate viewing window.

    As seen on the graph, this graph has no x-intercepts. Therefore, the equation

$0.5x^2 + 1.35x + 4.2 = 0$ has no real solutions. It will have complex solutions. If you wanted to find the complex solutions, we would need to use the quadratic formula.

Equations Reducible to Quadratic Form

Some equations can be solved with the techniques of this section although the equation is not quadratic. This process involves making a substitution so that the new equation is quadratic. Such equations that can be rewritten are said to be quadratic in form. Be sure to solve the original equation once you have completed solving the quadratic equation. The substitution usually involves the variable u.

Examples:

1. Solve $x^4 - 3x^2 - 10 = 0$.

   Looking at this equation, if we substitute $u = x^2$, the equation becomes $u^2 - 3u - 10 = 0$. This is a quadratic equation which can be solved by factoring.

   $u^2 - 3u - 10 = 0$

   $(u - 5)(u + 2) = 0$

   $u = 5$ or $u = -2$

   Now, remember that $u = x^2$. This gives $x^2 = 5$ and $x^2 = -2$. Solving for x, we find that $x = \pm\sqrt{5}$ or $x = \pm\sqrt{2}i$.

2. Solve $x + 4\sqrt{x} - 1 = 0$.

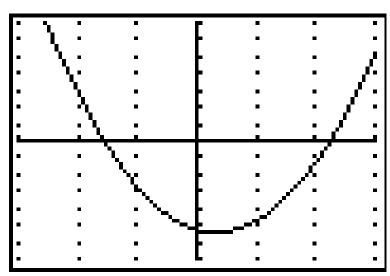
   If we choose $u = \sqrt{x}$, then this equation becomes $u^2 + 4u - 1 = 0$. This is a quadratic equation that can be solved with the quadratic formula.

   $u = \frac{-4\pm\sqrt{4^2-4(1)(-1)}}{2(1)} = \frac{-4\pm\sqrt{20}}{2} = -2 \pm \sqrt{5}$

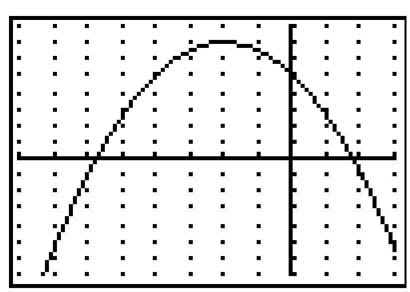
   Now, remember that $u = \sqrt{x}$, so $\sqrt{x} = -2 \pm \sqrt{5}$ or $x = (-2 \pm \sqrt{5})^2$.

Homework:

1. Estimate the solutions to $1.5x^2 - x - 5.25 = 0$ using the graph of $y = 1.5x^2 - x - 5.25$ shown.



2. Estimate the solutions to $-0.5x^2 - 2x + 5 = 0$ using the graph of $y = -0.5x^2 - 2x + 5$ shown.



3. Solve graphically using your graphing calculator $-3x^2 + 4x + 10 = 0$.

4. Solve graphically using your graphing calculator $0.2x^2 - 5x + 10 = 0$.

5. Solve graphically using your graphing calculator $1.2x^2 + 4x – 7.5 = 0$.

6. Solve graphically using your graphing calculator $-2x^2 – x + 8 = 0$.

7. Solve $x^2 + 9x + 20 = 0$ numerically given the table of $y_1 = x^2 + 9x + 20$ as shown.

X	Y_1	
-6	2	
-5	0	
-4	0	
-3	2	
-2	6	
-1	12	
0	20	

Press + for ΔTbl

8. Solve $x^2 + x – 2 = 0$ numerically given the table of $y_1 = x^2 + x – 2$ as shown.

X	Y_1	
-2	0	
-1	-2	
0	-2	
1	0	
2	4	
3	10	
4	18	

X=4

9. Solve $x^2 + 4x – 12 = 0$ numerically. Show a table.

10. Solve $2x^2 – 18 = 0$ numerically. Show a table.

Solve the following by taking square roots.

11. $6x^2 + 7 = 19$

12. $4x^2 – 3 = 11$

13. $5(x + 1)^2 – 2 = 18$

14. $2(3x – 1)^2 + 1 = 21$

Solve the following by factoring.

15. $6x^2 – 13x – 5 = 0$

16. $3x^2 – 14x – 5 = 0$

17. $x^2 – 12x + 35 = 0$

18. $6x^2 + 46x + 28 = 0$

Solve by completing the square.

19. $x^2 - 16x - 40 = 0$

20. $x^2 + 5x + 15 = 0$

21. $4x^2 + 24x + 13 = 0$

22. $6x^2 - 13x - 5 = 0$

Solve the following by the quadratic formula.

23. $4x^2 - 2x - 3 = 0$

24. $2x^2 + 5x + 7 = 0$

25. $-3x^2 - 7x + 9 = 0$

26. $x^2 + 9x - 11 = 0$

Solve by any method.

27. $7(2x + 1)^2 - 9 = 5$

28. $3x^2 - 17 = 0$

29. $x^2 - 8x + 7 = 0$

30. $3x^2 - 3x + 2 = 0$

31. $4x^2 - 9 = 0$

32. $5x^2 - 9x + 3 = 0$

Solve the following.

33. $2x^4 - 9x^2 + 4 = 0$

34. $3x^4 + x^2 - 5 = 0$

35. $2x - 7\sqrt{x} + 3 = 0$

36. $x - 5\sqrt{x} - 2 = 0$

6.3 Graphing Quadratic Functions

In the previous section, we saw that there are several ways to solve a quadratic equation of the form $ax^2 + bx + c = 0$. We also saw when solving these equations graphically, that the solutions were the x-intercepts of the graph. We can use these ideas to help graph equations of quadratic functions and also to write the equation of quadratic functions.

Recall when sketching complete graphs, the graphs should include the intercepts if the intercepts make sense in context of the problem. We can find the intercepts of quadratic graphs using the rules we have used for finding horizontal and vertical intercepts in previous sections. Given the equation $f(x) = x^2 + 5x + 4$, the vertical intercept is found by setting x =0 in the equation. This gives a vertical intercept of (0, 4). The horizontal intercepts are found by setting f(x) = 0 in the equation. This gives $x^2 + 5x + 4 = 0$ which can be solved by factoring (or the quadratic formula). Solving gives x = -4 or x = -1. So the horizontal intercepts are (-4, 0) and (-1, 0).

Examples:

Find the horizontal and vertical intercepts of:

1. $f(x) = 6x^2 + 15x + 6$
2. $g(x) = 2x^2 – 5x – 12$
3. $h(x) = x^2 + 8$

Solutions:

1. To find the vertical intercept, set x = 0:

   $f(0) = 6(0)^2 + 15(0) + 6 = 6$ so the vertical intercept is (0, 6).

   To find the horizontal intercepts, set the equation equal to 0.

   $6x^2 + 15x + 6 = 0$

   $3(2x^2 + 5x + 2) = 0$

   $3(2x + 1)(x + 2) = 0$

   $2x + 1 = 0$ or $x + 2 = 0$

   $x = -½$ or $x = -2$

   The horizontal intercepts are (- ½ , 0) and (-2, 0).

2. To find the vertical intercept:

   $g(0) = 2(0)^2 – 5(0) – 12 = -12$ so the vertical intercept is (0, -12).

   To find the horizontal intercepts, set the equation equal to 0.

   $2x^2 – 5x – 12 = 0$

   $(2x + 3)(x – 4) = 0$

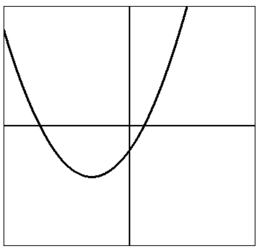
   $2x + 3 = 0$ or $x – 4 = 0$

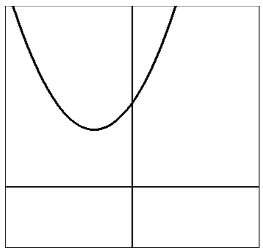
   $x = -3/2$ or $x = 4$

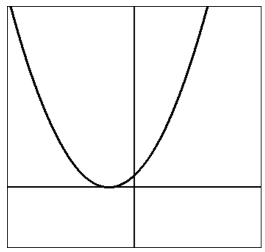
   The horizontal intercepts are (-3/2, 0) and (4, 0).

3. To find the vertical intercept, set x = 0.
   $h(0) = 0^2 + 8 = 8$ so the vertical intercept is (0, 8).
   To find the horizontal intercepts, set the equation equal to 0.
   $x^2 + 8 = 0$
   $x^2 = -8$
   $x = \pm\sqrt{-8}$ This equation has no real solutions.
   Therefore, the graph has no horizontal intercepts.

All quadratic equations will have a vertical intercept. They may have either 0, 1, or 2 horizontal intercepts as illustrated in the graphs shown below.


2 x-intercepts


0 x-intercepts

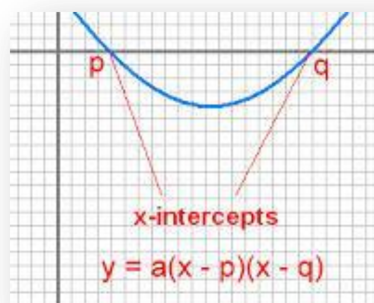

1 x-intercept

Graphing Quadratic Functions in Intercept Form

In general, the **intercept form of a quadratic equation** is:

$$y = a(x-p)(x-q)$$

where (p, 0) and (q, 0) are the x-intercepts of the graph. The value of a determines the width of the parabola and its concavity. The intercept form of a quadratic equation is the form of a quadratic equation by which you can easily tell the x intercepts of the quadratic equation.

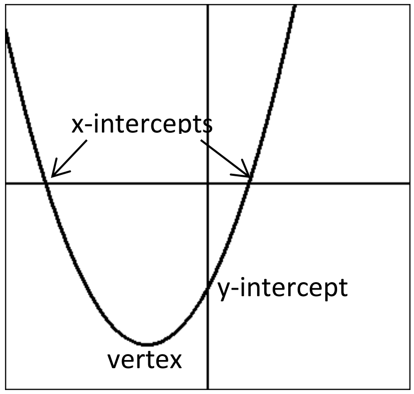


Example:

The x-intercepts of parabolas can be obtained from their quadratic equations in intercept forms as follows:

Quadratic Equation in intercept form	x-intercepts of its parabola
$y = (x - 3)(x - 4)$	(3, 0) and (4, 0)
$y = 3x(x - 5)$	(0, 0) and (5, 0)
$y = (x + 3)(x + 5)$	(-3, 0) and (-5, 0)

To graph a complete graph of a quadratic function, we should show the intercepts and the vertex of the parabola.

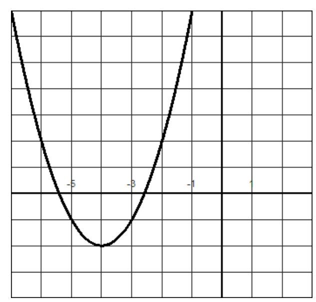


Graphing Quadratic Functions in Vertex Form

Vertex form is $f(x) = a(x – h)^2 + k$. Recall that the graph of $f(x) = a(x – h)^2 + k$ is a transformation of the graph of $f(x) = x^2$. The value of **a** determines the width and concavity of the graph. The value of **h** gives the horizontal shift and the value of **k** gives the vertical shift of the graph.

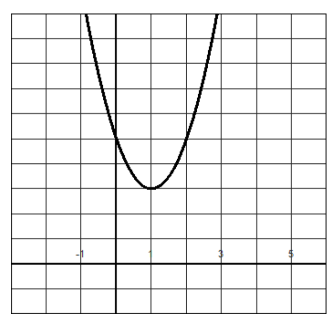
Examples:

1. Graph $f(x) = (x + 4)^2 – 2$ and identify the vertex.

   Solution:
   This graph will be shifted 4 units to the left and 2 units downward. The vertex will be (-4, -2).

2. Graph $g(x) = 2(x – 1)^2 + 3$ and identify the vertex of the graph.

   Solution:
   This graph will be shifted 1 unit to the right and 3 units upward. The graph is stretched vertically by 2. The vertex is (1, 3).

The vertex of a quadratic function in vertex form $f(x) = a(x – h)^2 + k$ is at the point (h, k).

For example, the vertex of $f(x) = -(x + 2)^2 + 5$ is at (-2, 5) and the vertex of $g(x) = 4(x – 7)^2 - 1$ is at the point (7, -1).

Recall from the discussion on transformations that the value of **a** affects the points on either side of the vertex.

We can now find all important points of the graph of a parabola when the equation is in vertex form.

Example:

Find the vertex and intercepts of $h(x) = 3(x + 1)^2 – 6$.

The y-intercept is found by setting x = 0. $h(0) = 3(0 + 1)^2 – 6 = 3 – 6 = -3$ so the y-intercept is (0, -3).

The x-intercepts are found by setting h(x) = 0.

$3(x + 1)^2 – 6 = 0$

$3(x + 1)^2 = 6$

$(x + 1)^2 = 2$

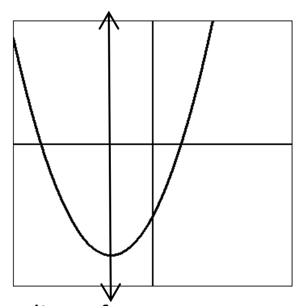
$x + 1 = \pm\sqrt{2}$

$x = -1 \pm \sqrt{2}$

So the x-intercepts are $(-1 + \sqrt{2}, 0)$ and $(-1 - \sqrt{2}, 0)$.

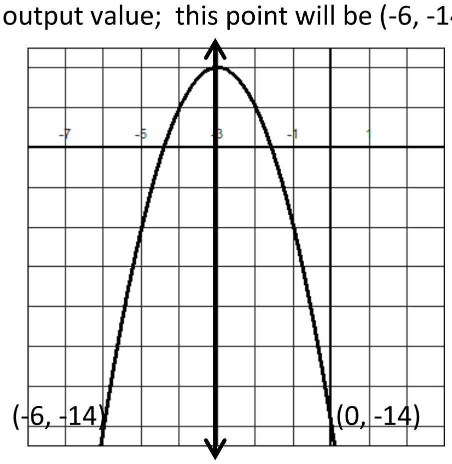
The vertex is (-1, -6). Note, the graph is stretched vertically by 3.

When sketching a parabola, we can use symmetry to find other points on the graph. Recall that the graph of any parabola is symmetric about a vertical line that goes through its vertex. That line is called the **line of symmetry.** If you find a point on one side of the line of symmetry, the graph must have a point equidistant on the other side of the line which has the same output value.



line of symmetry

For example, $f(x) = -2(x + 3)^2 + 4$ has a vertex at (-3, 4). The line of symmetry for this graph is x = -3. We can also find the y-intercept which is (0, -14). This point is three units to the right of the line of symmetry. A point symmetric to this one will be three units to the left of the line of symmetry and have the same output value; this point will be (-6, -14).



line of symmetry

x = -3

Graphing Quadratic Functions in Standard form

A quadratic function in standard form is $f(x) = ax^2 + bx + c$. First, let's investigate how the values of **a**, **b**, and **c** affect the graph of the parabola.

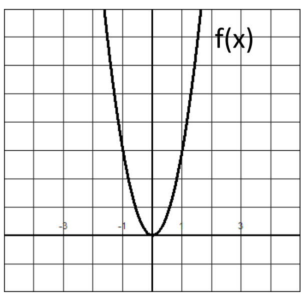
Previously, when discussing transformations, we found that the value of **a** affects the width of the parabola and its concavity. If $|a| > 1$, then the graph is stretched vertically. If $0 < |a| < 1$, then the graph is shrunk vertically. If $a > 0$, then the graph is concave up (opens upward) and if $a < 0$, then the graph is concave down (opens downward).

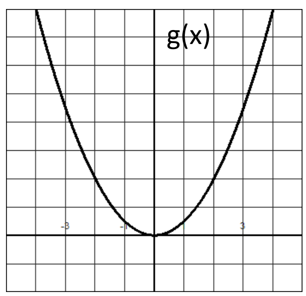
Example:

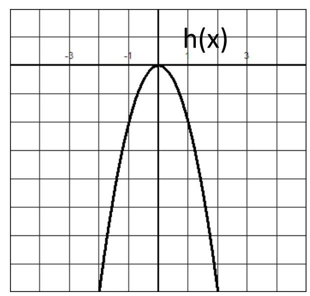
Given the equations $f(x) = 3x^2$ and $g(x) = ½ x^2$, and $h(x) = -2x^2$, describe the graphs and sketch.

Solution:

$f(x) = 3x^2$ is stretched vertically by 3 and is concave up
$g(x) = ½ x^2$ is shrunk vertically by ½ and is concave up
$h(x) = -2x^2$ is stretched vertically by 2 and is concave down







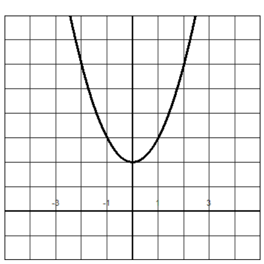
The value of **c** determines the y-intercept of the graph. To find a y-intercept, we set x = 0. So, $f(0) = a(0)^2 + b(0) + c = c$. The point (0, c) is the y-intercept of the graph. For example, $f(x) = 4x^2 – 6x + 1$ has a y-intercept at (0, 1) and $g(x) = 2x^2 – 5$ has a y-intercept at (0, -5).

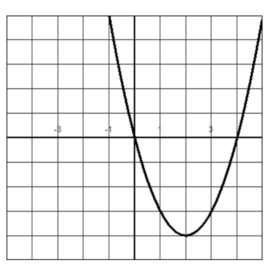
The coefficient **b** affects the vertex of the parabola. If b = 0, then the vertex of the parabola will be on the y-axis. If b ≠ 0, then the vertex is shifted off of the y-axis.

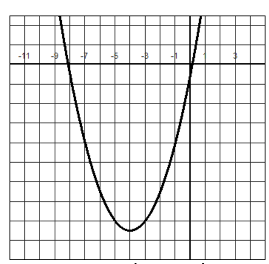
Example:

Sketch the following graphs and identify the vertex.
$f(x) = x^2 + 2$, $g(x) = x^2 – 4x$, $h(x) = x^2 + 8x – 1$

Solution:


vertex: (0, 2)


vertex: (2, -4)


vertex: (-4, -17)

Notice in the above example that the first graph has a vertex on the y-axis at (0, 2) and the other two graphs have a vertex that is not on the y-axis. The vertex on these graphs has not just been shifted right or left though. We will find a formula for the vertex later in this section.

Therefore, given an equation in standard form we should be able to identify the concavity, the width, the y-intercept and whether or not the vertex is shifted off the y-axis.

Examples:

Given the following equations, describe what you know about the graphs based on the values of **a**, **b**, and **c**.

1. $f(x) = 2x^2 - 3x + 7$
2. $g(x) = -\frac{1}{2}x^2 - 5$
3. $h(x) = \frac{2}{3}x^2 + 2x$

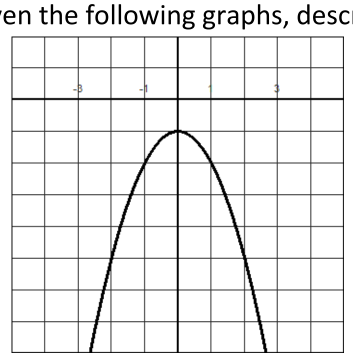
Solution:

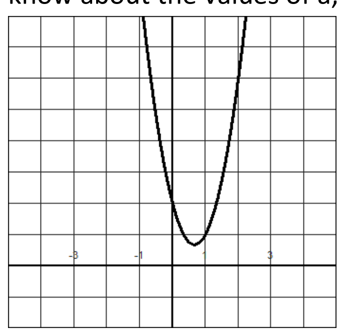
1. f(x) is concave up ($a > 0$) and stretched ($|a| > 1$). The y-intercept is at (0, 7) and the vertex will not be on the y-axis ($b \neq 0$).
2. g(x) is concave down ($a < 0$) and shrunk ($|a| < 1$). The y-intercept is at (0, -5) and the vertex will be on the y-axis ($b = 0$).
3. h(x) is concave up ($a > 0$) and shrunk ($|a| < 1$). The y-intercept is at (0, 0) and the vertex will not be on the y-axis ($b \neq 0$).

Given a graph, we should be able to describe the appropriate values for **a**, **b**, and **c**.

Examples:

Given the following graphs, describe what you know about the values of a, b, and c.

1. 

2. 

Solutions:

1. Since the graph is concave down, $a < 0$. The vertex is on the y-axis so $b = 0$. The y-intercept is below the x-axis so $c < 0$.
2. The graph is concave up so $a > 0$. The vertex is not on the y-axis so $b \neq 0$. The y-intercept is above the x-axis so $c > 0$.

To find the vertex of a parabola when the equation is in standard form, we use the formula x = - $\frac{b}{2a}$ to find the x-coordinate of the vertex. Then, we use the equation to find the value of f(x).

Examples:

Find the vertex of the following functions.

1. $f(x) = 2x^2 + 12x - 5$
2. $g(x) = -x^2 + 5x + 2$

Solutions:

1. The x-value of the vertex is at x = - $\frac{b}{2a}$. For this equation, a = 2 and b = 12 so the x-value of the vertex is x = $-\frac{12}{2(2)} = -3$.
   Use the equation to find the y-value of the vertex. $f(-3) = 2(-3)^2 + 12(-3) - 5 = 18 - 36 - 5 = -23$.
   The vertex is (-3, -23).

2. For this equation, a = -1 and b = 5 so the x-value of the vertex is x = $-\frac{5}{2(-1)} = \frac{5}{2}$.
   Use the equation to find the y-value of the vertex. $g(\frac{5}{2}) = -(\frac{5}{2})^2 + 5(\frac{5}{2}) + 2 = \frac{33}{4}$.
   The vertex is $(\frac{5}{2}, \frac{33}{4})$.

Previously, we found the intercepts of parabolic functions written in standard form. Using the vertex and the intercepts, we can sketch a complete graph.

Example:

Sketch a complete graph of $f(x) = 3x^2 - 6x - 2$.

Solution:

The y-intercept is (0, -2). We also know the graph should be concave up and stretched by 3.

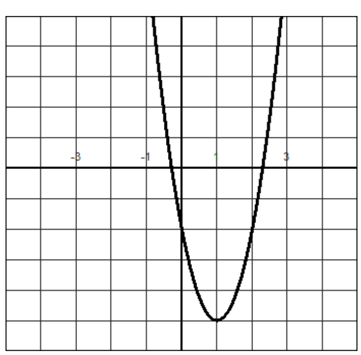
The x-intercepts can be found using the quadratic formula.

x = $\frac{-(-6)\pm\sqrt{(-6)^2-4(3)(-2)}}{2(3)} = \frac{6 \pm \sqrt{60}}{6}$ so the x-intercepts are (-0.29, 0) and (2.29, 0).

The vertex is:

x = $-\frac{-6}{2(3)}$ = 1, $f(1) = 3(1)^2 - 6(1) - 2 = -5$ so the vertex is (1, -5).

The graph is:



Rewriting Equations from Standard Form to Vertex Form

To change an equation from standard form to vertex form, we use a technique called completing the square.

Steps to complete the square:

1. Put the terms in descending order. Group the first two terms.
2. If the leading coefficient is not 1, factor the leading coefficient from the first two terms.
3. Using $x^2 + bx$, complete the square by taking half of b and squaring the result. Add and subtract this value. If the leading coefficient is not 1, you will need to multiply one of these by the coefficient to separate it from the perfect square trinomial.
4. Write $x^2 + bx + c$ as $(x + b/2)^2$.
5. Collect like terms.

Examples:

1. Rewrite $f(x) = x^2 + 8x - 1$ in vertex form and identify the vertex.

$f(x) = (x^2 + 8x) - 1$	Group the first two terms. Leading coefficient is 1.
$f(x) = (x^2 + 8x + 16) - 16 - 1$	Since b = 8, then $(8/2)^2 = 16$ so we add and subtract 16.
$f(x) = (x + 4)^2 - 17$	Factor $x^2 + 8x + 16$ into $(x + 4)^2$.

The equation is now in vertex form. The parabola has a vertex of (-4, -17).

2. Rewrite $f(x) = 2x^2 - 10x + 8$ in vertex form and identify the vertex.

$f(x) = (2x^2 - 10x) + 8$	Group the first two terms. The leading coefficient is 2, so factor.
$f(x) = 2(x^2 - 5x) + 8$	Factor the 2 out of the first two terms.
$f(x) = 2(x^2 - 5x + 25/4 - 25/4) + 8$	Since b = 5, then $(5/2)^2 = 25/4$, so add and subtract 25/4.
$f(x) = 2(x - 5/2)^2 - 25/2 + 8$	Factor $x^2 - 5x + 25/4$ into $(x - 5/2)^2$. Multiply -25/4 by the coefficient of 2.
$f(x) = 2(x - 5/2)^2 - 9/2$	Collect like terms of -25/2 and 8.

The equation is now in vertex form. The parabola has a vertex of (5/2, -9/2).

Homework:

Find the vertical and horizontal intercepts of the following if they exist.

1. $f(x) = x^2 – 25$
2. $F(x) = x^2 + 11x + 28$
3. $g(x) = 2x^2 + x – 3$
4. $Y = 2x^2 – 7x + 5$
5. $h(x) = 12x^2 – 32x + 20$
6. $y = 2(x + 6)^2 – 32$
7. $f(x) = x^2 + 3$
8. $F(x) = 2x^2 – 7x + 1$
9. $h(x) = -x^2 + 3x – 11$
10. $y = 4(x – 2)^2 + 6$

Given the equations, identify the vertex and the line of symmetry.

11. $f(x) = (x + 7)^2 – 2$
12. $g(x) = (x + 5)^2 + 3$
13. $f(x) = 2(x - 3)^2 – 1$
14. $h(x) = -(x + 1)^2 – 9$

Given the equations, identify all intercepts and the vertex of the graph.

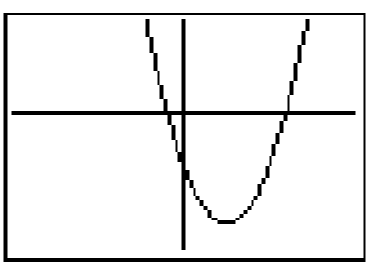
15. $f(x) = -2(x + 6)^2 – 2$
16. $h(x) = 3(x + 2)^2 – 9$
17. $g(x) = 1/3\ (x - 3)^2 – 4$
18. $f(x) = -(x + 5)^2 – 8$

Given the equations, sketch the graph and identify the vertex and the intercepts of the graph.

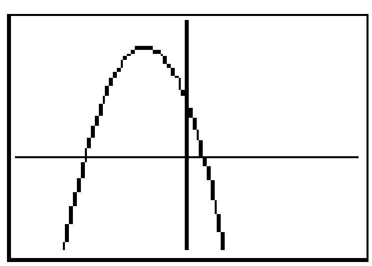
19. $f(x) = (x + 5)^2 + 2$
20. $g(x) = 2(x - 7)^2 – 8$
21. $f(x) =- ½\ (x - 1)^2 – 5$
22. $f(x) = -(x + 3)^2 – 9$

For the following graphs, identify whether the values of a and c are positive, negative, or zero. Identify whether b is equal to zero or not equal to zero.

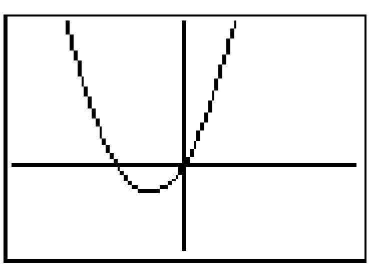
23.



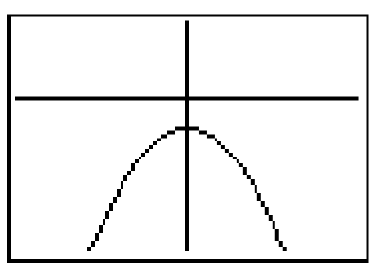
24.



25.



26.



Given the following equations, describe what you know about the graphs based on the values of a, b, and c.

27. $f(x) = -4x^2 - 3x + 11$

28. $g(x) = 6x^2 + 1$

29. $h(x) = \frac{2}{3}x^2 + 2x - 5$

30. $h(x) = -\frac{1}{2}x^2 + x + 3$

Given the following equations, identify the vertex and the intercepts and sketch the graph.

31. $f(x) = 4x^2 - 5$

32. $g(x) = 6x^2 + x - 5$

33. $h(x) = \frac{1}{3}x^2 + 2x$

34. $f(x) = -x^2 + 4x + 3$

35. Given $y = -x^2 + 3x + 4$, find:
   - A. the y-intercept.
   - B. the x-intercepts.
   - C. the vertex.
   - D. Is the graph concave up or down?
   - E. Is the graph stretched, shrunk, or standard width?
   - F. Sketch the graph.

36. Given $y = 2x^2 - 8x + 4$, find:
   - A. the y-intercept.
   - B. the x-intercepts.
   - C. the vertex.
   - D. Is the graph concave up or down?
   - E. Is the graph stretched, shrunk, or standard width?
   - F. Sketch the graph.

37. Given $y = \frac{1}{2}x^2 + 3x - 1$, find:
   - A. the y-intercept.
   - B. the x-intercepts.
   - C. the vertex.
   - D. Is the graph concave up or down?
   - E. Is the graph stretched, shrunk, or standard width?
   - F. Sketch the graph.

Rewrite the following in vertex form.

38. $f(x) = x^2 + 12x + 5$

39. $f(x) = x^2 + 7x - 10$

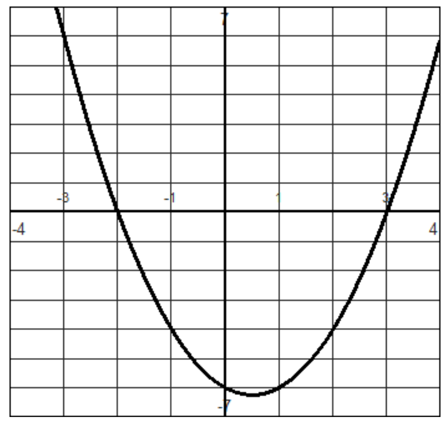
40. $f(x) = 3x^2 - 12x + 17$

41. $f(x) = 2x^2 + 20x - 11$

6.4 Writing Equations of Quadratic Functions

Quadratic equations can be written in three ways. One way is to use the vertex and one point on the graph and write the equation using the vertex form of the equation. Another way is to use the x-intercepts and one other point on the graph and write the equation in standard form. The third way is to use any three points on the graph and use the elimination method to solve a system of equations to find the equation in standard form.

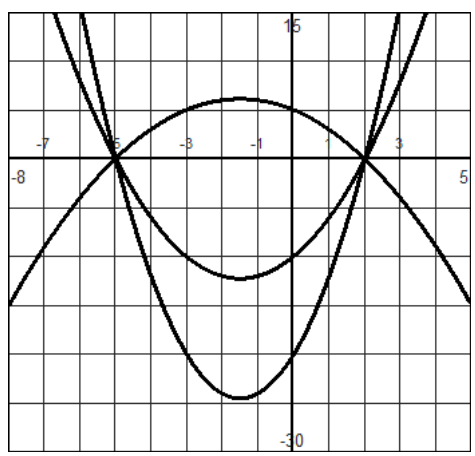
We can write an equation of a quadratic function easily if we can identify the x-intercepts of the graph. This equation can be derived by reversing the steps we used to solve by factoring. Look at the graph shown to the right.



Let us assume that the graph has not been stretched or shrunk. Therefore, we will assume a = 1. This graph has x-intercepts at (-2, 0) and (3, 0). Therefore, when solving by factoring we would have gotten solutions of
x = -2 and x = 3.
Rewriting these equations so they are set equal to zero gives (x + 2) =0 and (x – 3) = 0. This means the quadratic equation in factored form was (x + 2)(x – 3) = 0.
If you would like the equation in standard form, multiply to get $x^2 - x - 6 = 0$. Therefore, the equation of the graph shown is $f(x) = x^2 - x - 6$.

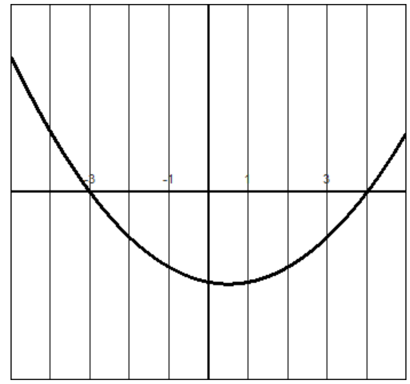
To realize there is more to the process to writing the equation than identifying the x-intercepts, let's look at the graphs of:

1. $y = x^2 + 3x - 10 = (x + 5)(x - 2)$
2. $y = 2(x^2 + 3x - 10) = 2(x + 5)(x - 2)$
3. $y = -\frac{1}{2}(x^2 + 3x - 10) = -\frac{1}{2}(x + 5)(x - 2)$



Notice that all of the graphs shown above have the same x-intercepts at (-5, 0) and (2, 0). The graphs vary in width and concavity. To write an equation of a graph, we need to look at more than the x-intercepts of the graph because the graph may have been stretched, shrunk, and/or reflected.

For example, given the graph shown to the right, we can see the x-intercepts are (-3, 0) and (4, 0). To write an equation for the function, we reverse the steps of solving using the zero-factor principle.

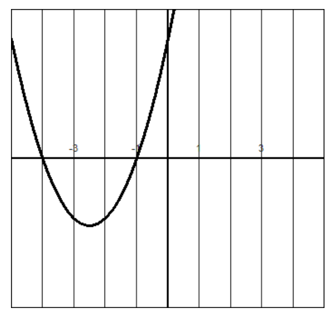


x = -3 and x = 4
x + 3 = 0 and x – 4 = 0
(x + 3)(x – 4) = 0
$x^2 – x – 12 = 0$

The equation of the graph shown could be $y = x^2 – x – 12$. But it could also be $y = a(x^2 – x – 12)$ where a is any positive number. We know a has to be a positive number because the graph is concave up but we do not know whether or not the graph has been stretched, shrunk, or is standard width. Without a scale on the y-axis, we cannot find the value of a for this exact graph. So, any equation in the form $y = a(x^2 – x – 12)$ is a possible equation for the graph shown.

Example:

Write two possible equations for the graph shown.

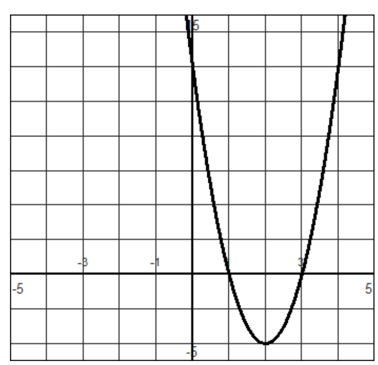


Solution:
The x-intercepts are (-4, 0) and (-1, 0).
x = -4 or x = -1
x + 4 = 0 or x+ 1 = 0
(x+ 4)(x+ 1) = 0
$x^2 + 5x + 4 = 0$
Two possible equations are $y = x^2 + 5x + 4$ or $y = 2(x^2 + 5x + 4)$.
Note that the value of the constant a was chosen to be 2 but it could be any positive number.

If there is a scale on the y-axis or if you can identify another point on the graph, we can find a value of a which gives the equation of the graph. Use the third point, plug the values of x and y into the equation and solve for a.

Examples:

1. Find the equation for the graph shown.



Solution:
The x-intercepts are (1, 0) and (3, 0).
x = 1 or x = 3
(x – 1)(x – 3) = 0
$y = a(x^2 – 4x + 3)$
Using the point (0, 12):
$12 = a(0^2 – 4(0) + 3)$ or 12 = 3a
a = 4 so the equation is $y = 4(x^2 – 4x + 3)$.

2. Find an equation for a graph which has intercepts at (6, 0), (9, 0) and (0, -2).

   Solution:
   Using the x-intercepts:
   x = 6 or x = 9
   (x – 6)(x – 9) = 0
   $x^2 - 15x + 54 = 0$
   So the equation must have the form $y = a(x^2 - 15x + 54)$. Plugging in the y-intercept gives:
   -2 = a (54)
   a = -2/54 = -1/27.
   The equation is $y = -1/27(x^2 - 15x + 54)$.

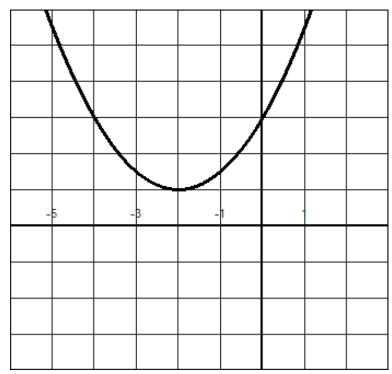
Writing Quadratic Functions in Vertex Form

Previously we learned how to write the equation of quadratics in vertex form given the graph.

Examples:

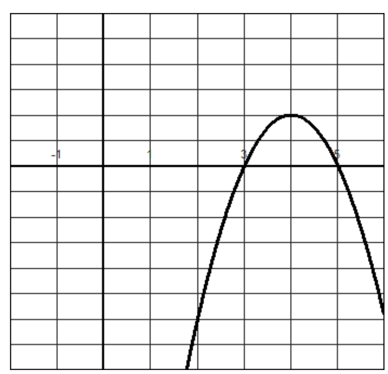
1. Write the equation of the graph shown.

   Solution:
   The vertex of the graph is (-2, 1). The two points which are one unit to the right and one unit to the left of the vertex are ½ unit up.
   Therefore, the equation is $f(x) = ½ (x + 2)^2 + 1$.



2. Write the equation of the graph shown.

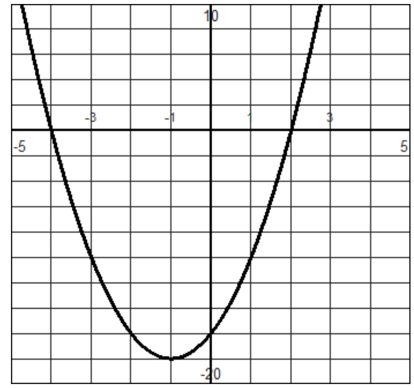
   Solution:
   The vertex of the graph is (4, 2). The two points which are one unit to the right and one unit to the left of the vertex are 2 units down.
   Therefore, the equation is $f(x) = -2 (x – 4)^2 + 2$.



Let's first look at examples of the two methods we have used previously using the same graph.

Example:

Given the graph shown, write the equation in A) vertex form and B)standard form.



Solution:

A) We can see from the graph that the vertex is (-1, -18). Therefore, in vertex form the equation will be $y = a(x + 1)^2 - 18$. Choosing another point, we can find the value of a. Let's choose (0, -16).

$-16 = a\,(0+1)^2 - 18$
$-16 = a - 18$
$a = 2$ so the equation is $y = 2(x + 1)^2 - 18$.

B) We can see from the graph that the x-intercepts are (0, -4) and (0, 2). Working backwards:

$x = -4$ and $x = 2$
$(x+ 4)(x - 2) = 0$
$X^2 + 2x - 8 = 0$
So the equation is $y = a(x^2 + 2x - 8)$. Choosing any other point, we can find the value of a. Let's choose (0, -16).

$-16 = a(-8)$ or $a = 2$. So the equation is $y = 2(x^2 + 2x - 8)$.

Writing a quadratic equation through any three points

Previously, we gathered data and used linear regression to find a line that approximated the data. We can find a quadratic function that approximates data if the data looks like it is quadratic in shape. We can use the elimination method to find a quadratic equation through any three points. The equation will be in standard form.

Steps to write the equation:

1. Identify any three points on the graph.
2. Use the equation $y = ax^2 + bx + c$ and the three points to write a system of three equations by plugging in the values of x and y from each of the three points.
3. Use the elimination method to eliminate c using 2 different pairs of equations.
4. Solve the new system of two equations for either a or b using elimination. Find the other variable.
5. Find the value for c.
6. Plug the values for a, b, and c into the standard form of the equation.
7. Check.

Example:

Write an equation of a quadratic through the points (1, 7), (3, 35) and (-2, 25).

Solution:

Using $y = ax^2 + bx + c$ and plugging in each of the three points:
$7 = a(1)^2 + b(1) + c$
$35 = a(3)^2 + b(3) + c$
$25 = a(-2)^2 + b(-2) + c$

This gives the three equations:
(1) $a + b + c = 7$
(2) $9a + 3b + c = 35$
(3) $4a – 2b + c = 25$

Choosing 2 pairs of equations and eliminating c:
We can multiply equation (1) by -1 and add to equation (2):
(1) $-a – b – c = -7$
(2) $\underline{9a + 3b + c = 35}$
(4) $8a + 2b = 28$

We can multiply equation (1) by -1 and add to equation (3):
(1) $-a – b – c = -7$
(3) $\underline{4a – 2b + c = 25}$
(5) $3a – 3b = 18$

Using equation (4) and (5) solve for a and b.
We can multiply equation (4) by 3 and equation (5) by 2 to eliminate b.
(4) $24a + 6b = 84$
(5) $\underline{6a – 6b = 36}$
$30a = 120$
$a = 4$

Plugging back into equation (4) to find b:
(4) $8(4) + 2b = 28$
$32 + 2b = 28$
$2b = -4$
$b = -2$

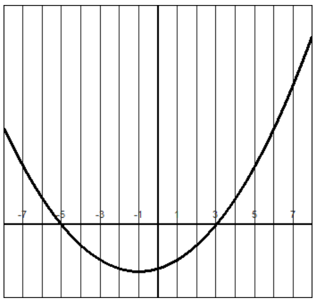
Plugging back into equation (1) to find c:
(1) $4 + (-2) + c = 7$
$c = 5$

The equation of the quadratic function is $y = 4x^2 - 2x + 5$. Verify that this equation is satisfied by the three given points.

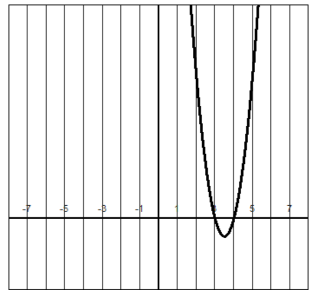
Homework:

Write 2 possible equations for the graphs shown.

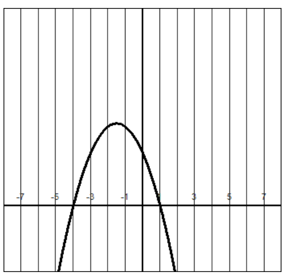
1.



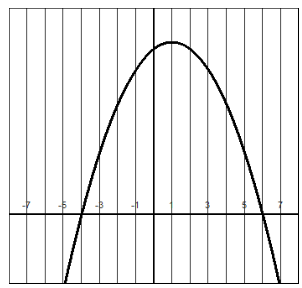
2.



3.



4.



5. Write the equation for a graph with the points (4, 0), (-7, 0) and (1, 3).

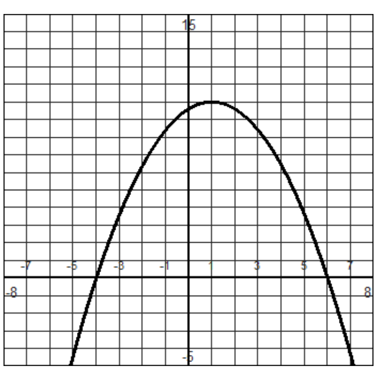
6. Write the equation for a graph with the points (-5, 0), (-2, 0) and (0, 3).

7. Write the equation for a graph with the points (-6, 0), (7, 0) and (0, -4).

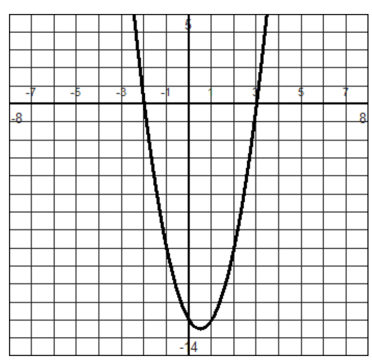
8. Write the equation for a graph with the points (1/2, 0), (-3, 0) and (4, 5).

Write equations for the graphs shown using the intercepts.

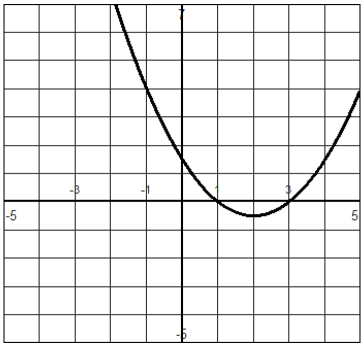
9.



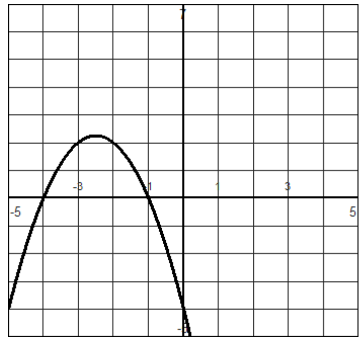
10.



11.

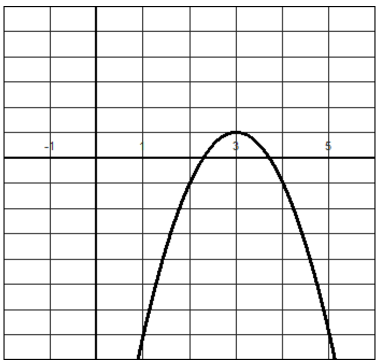


12.

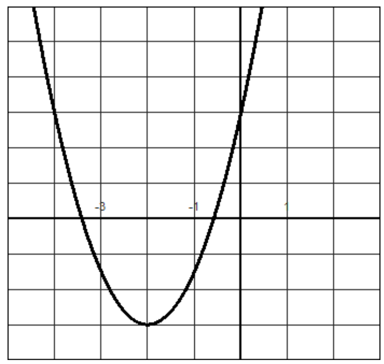


Write the equation of the graph shown in vertex form.

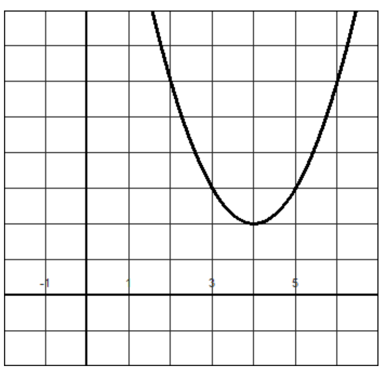
13.



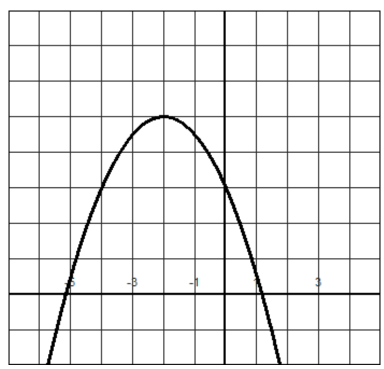
14.



15.

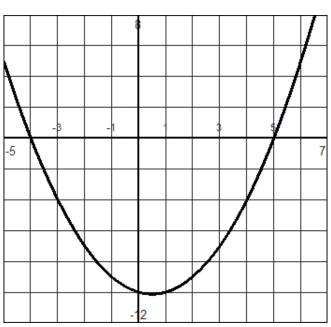


16.

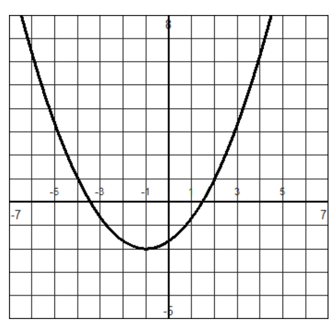


Find equations of the following graphs.

17.



18.



19. Write the equation of the parabola that has a vertex at (1/2, -5) and the point (2, 7).

20. Write the equation of the parabola that has a vertex at (-3, -5) and the point (1, -8).

21. Write the equation of the parabola that has the points (3, 0), (5, 0) and (0, 10).

22. Write the equation of the parabola that has the points (-2, 0), (4, 0) and (-1, 2).

23. Write the equation of the parabola that has the points (-2, 14), (2, 10), and (3, 19).

24. Write the equation of the parabola that has the points (-4, 5), (2, 7), and (6, 35).

25. Write the equation of the parabola that has the points (-3, -13), (-1, 0), and (1, 1).

26. At 1821 feet tall, the CN Tower in Toronto, Ontario, is the world's tallest self-supporting structure. Suppose you are standing in the observation deck on top of the tower and you drop a penny from there and watch it fall to the ground. The table shows the penny's distance from the ground after various periods of time (in seconds) have passed. Write an equation for the height of the penny at any time t.

Time (seconds)	Distance (feet)
0	1821
2	1757
6	1245
10	221

27. The table lists the number of Americans (in thousands) who are expected to be over 100 years old for selected years. [Source: US Census Bureau.] Write an equation for the number of Americans in thousands who are expected to be over 100 years old where t is time in years after 1990. How many Americans will be over 100 years old in the year 2008?

Year	Number (thousands)
1994	50
1998	65
2000	75
2002	94
2004	110

28. The table shows the cost of driving a car at different speeds. The speeds, V, are given in miles per hour and the cost, C, includes fuel and maintenance for driving the car 100 miles at that speed.

v	30	40	50	60	70
C	6.5	6	6.2	7.8	10.6

29. The batter in a softball game hits the ball when it is 3.5 feet above the ground. The ball reaches its highest trajectory of 38 feet when it is 150 feet from home plate. Write an equation for the height of the softball in terms of its horizontal distance from home plate.

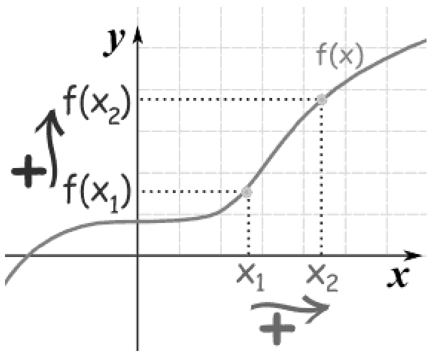
30. An acrobat is catapulted into the air from a springboard at ground level. Her maximum height of 11.025 meters is achieved after 1.5 seconds. Write an equation for the path of the acrobat.

31. Sam is standing 30 feet away from a basket and shoots the ball at the basket from a height of 6.5 feet. The hoop is 10 feet high. If the ball reaches a maximum height of 11.5 feet when it is 1 foot away from the basket, write an equation for the trajectory of the ball.

6.5 Increasing/Decreasing, Maximum/Minimum, and Applications of Quadratics

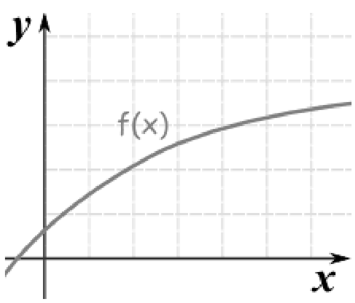
Increasing and Decreasing Functions

A function is increasing if the **outputs** increase as the **inputs** increase, like this:



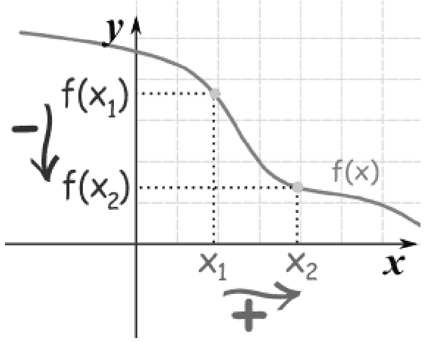
We can see that an increasing graph will rise from left to right. Mathematically, f(x) is increasing if for every $x_1 < x_2$, then $f(x_1) < f(x_2)$. This has to be true for **any** x_1, x_2, not just some input values. The rate of increase does not have to be constant.

Example:



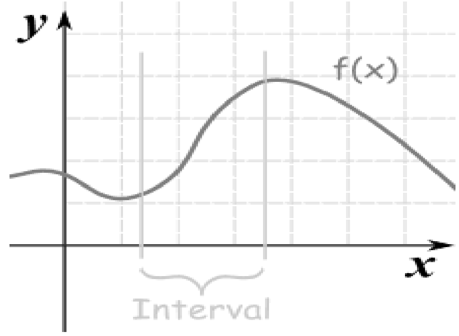
This is an increasing function whose rate of increase slows from left to right.

A function is decreasing if the **outputs** decrease as the **inputs** increase, like this:



We can see that a decreasing graph will fall from left to right. Mathematically, f(x) is decreasing if for every $x_1 < x_2$, then $f(x_1) > f(x_2)$. This has to be true for **any** x_1, x_2, not just some input values. The rate of decrease does not have to be constant.

Usually we will be interested in **some interval**, like this one which is increasing. The function is increasing on the interval shown (it may be increasing or decreasing elsewhere).



For a quadratic function, the graph changes from increasing to decreasing (for a concave down parabola) or from decreasing to increasing (for a concave up parabola) at the vertex. The vertex is called a **turning point** of the graph. For other graphs, the turning points may not be as easy to locate. These turning points occur at either maximum or minimum values of the function. There are two types of maximum and minimum values. These are called **global maximum or minimum values** when the values are the absolute highest or lowest on the entire domain of the function. If the maximum or minimum value is just higher or lower than the surrounding points on the graph, then these are called **local maximum or minimum values**.

Examples:

1. For $f(x) = 2(x - 3)^2 + 5$, identify the intervals where the graph is increasing and decreasing. Identify the maximum or minimum value of the graph.

   Solution:

   This graph is concave up because the value of a is positive. Since the equation is in vertex form, we can see that the vertex is at (3, 5). The vertex will be the lowest point on the graph; therefore, the minimum value is 5. The graph will be decreasing on the interval $(-\infty, 3)$ and increasing on the interval $(3, \infty)$.

2. For $f(x) = -x^2 + 4x – 7$, identify the intervals where the graph is increasing and decreasing. Identify the maximum or minimum value of the graph.

   Solution:

   This graph is concave down because the value of a is negative. We can find the vertex by using the formula $x = -\frac{b}{2a} = -\frac{4}{2(-1)} = 2$. Plugging x=2 back into the equation, we find the vertex to be (2, -3). The vertex will be the highest point on the graph; therefore, themaximum value is -3. The graph will be increasing on the interval $(-\infty, 2)$ and decreasing on the interval $(2, \infty)$.

Many of our applications in this chapter will revolve around minimum and maximum values of quadratic functions. Quadratic equations can appear in applications such as revenue, stopping distance of a car, height of a dropped or thrown object, and area problems.

When setting up the equation you should:

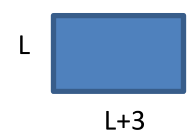
1. Write down given information. Determine what the variable represents.
2. Sketch a diagram if appropriate.
3. Identify any needed formulas.
4. Write the equation.
5. Solve using factoring, taking square roots, or the quadratic formula.
6. Verify the solutions and check the suitability of the solutions.

Examples:

1. We are going to fence in a rectangular field and we know that for some reason we want the field to have an enclosed area of 75 ft^2. We also know that we want the width of the field to be 3 feet longer than the length of the field. What are the dimensions of the field?

   Solution:

   If we let L be the length of the field, then we know that the width which is 3 feet longer than the length must be L+3. We are given that the area is 75 ft^2.

   Next, we know the area of a rectangle is its length times width.

   $A = L \cdot W$
   $75 = L(L + 3)$
   $75 = L^2 + 3L$
   $L^2 + 3L - 75 = 0$

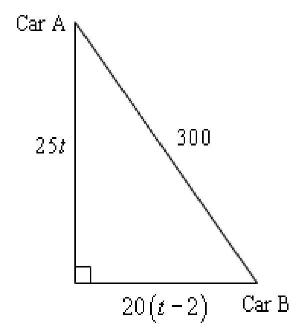
   Using the quadratic formula, we find $L = \frac{-3 \pm \sqrt{309}}{2}$. In decimal form, the two solutions to this equation are L = 7.289 and L = -10.289. As the length of a rectangle cannot be negative, the only reasonable solution is L = 7.289. Therefore, the length is 7.289 feet and the width will be 10.289 feet.

2. Two cars start out at the same point. One car starts out driving north at 25 mph. Two hours later, the second car starts driving east at 20 mph. How long after the first car starts traveling does it take for the two cars to be 300 miles apart?

Solution:

We'll start off by letting *t* be the amount of time that the first car, let's call it car A, travels. Since the second car, let's call that car B, starts out two hours later then we know that it will travel for t-2 hours.

Now, we know that the distance traveled by an object is its rate of travel (speed) times time traveled. So we have the following distances traveled for each car.
Distance of car A: 25t
Distance of car B: 20(t – 2)



Sketching a picture of the situation, we can see that we have a right triangle. That means that we can use the Pythagorean Theorem.

$(25t)^2 + (20(t - 2))^2 = 300^2$

Multiplying and simplifying gives:

$1025t^2 - 1600t - 88400 = 0$

Using the quadratic formula, we find t = $\frac{1600 \pm \sqrt{365000000}}{2050}$. Estimating the solutions, we get t = 10.1 and t = -8.54. The negative solution does not make sense because we are discussing time. Therefore, the time after the first car starts traveling for the two cars to be 300 miles apart is 10.1 hours.

3. The height,h , in feet of an object above the ground is given by h = $-16t^2 + 64t + 190$, where *t* is the time in seconds. Find the time it takes the object to strike the ground and find the maximum height of the object.

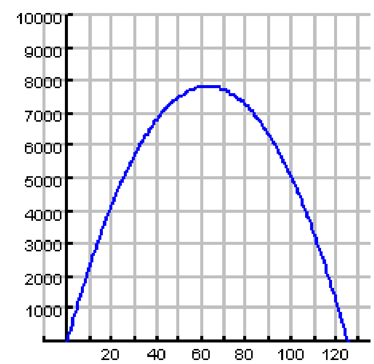
Solution:

The ground is 0 feet high. So, we need to solve $-16t^2 + 64t + 190 = 0$.
Using the quadratic formula we find t = $\frac{8 \pm \sqrt{254}}{4}$ or t = 5.98 and t = -1.98. As negative time is not reasonable, the time it takes for the object to hit the ground is 5.98 seconds.

The other part of the question is we want to know that maximum height that the object reaches. Since we can see that the function is clearly a quadratic function which opens down, we know that this maximum must occur at the vertex. So let's find the vertex. We use the formula $t = \frac{-b}{2a}$. So we have t = $\frac{-64}{2(16)}$ = 2 seconds.

So at 2 seconds the object reaches its maximum height. However, we wanted to know what that maximum height is. Therefore, we must find the value of the vertex, in this case it will be the value of h when t = 2. So we plug this in to the equation and find h = 254 feet. So the maximum height is 254 feet.

4. A parking lot is rectangular in shape and measures 250 feet around three of the four sides. Assuming the width of the parking lot is x, then the area of the lot can be modeled by A = (250-2x)x. A graph of the function is shown to the right. For what widths is the area increasing? For what widths is the area decreasing? For what width does the lot have its maximum area?



Solution:

As the area $A = 250x - 2x^2$, we can find that the vertex is at (62.5, 7812.5) using the vertex formula. The area of the parking lot is increasing on the interval [0, 62.5) and decreasing on the interval (62.5, 125]. Notice that the domain of this function is [0, 125]. The lot will have its maximum area of 7812.5 feet when the width of the lot is 62.5 feet.

Homework:

1. For $f(x) = -(x - 7)^2 + 5$, identify the intervals where the graph is increasing and decreasing. Identify the maximum or minimum value of the graph.

2. For $f(x) = (x + 11)^2 - 8$, identify the intervals where the graph is increasing and decreasing. Identify the maximum or minimum value of the graph.

3. For $f(x) = 3x^2 - 5x + 10$, identify the intervals where the graph is increasing and decreasing. Identify the maximum or minimum value of the graph.

4. For $f(x) = -2x^2 - 7x + 1$, identify the intervals where the graph is increasing and decreasing. Identify the maximum or minimum value of the graph.

5. We are going to fence in a rectangular field and we know that we want the field to have an enclosed area of 90 ft^2. We also know that we want the width of the field to be 5 feet longer than the length of the field. What are the dimensions of the field?

6. Two cars start out at the same point. One car starts out driving south at 35 mph. Three hours later the second car starts driving east at 25 mph. How long after the first car starts traveling does it take for the two cars to be 400 miles apart?

7. An object is dropped off of a 275-foot building and its height at time t, in seconds, is given by $h = -16t^2 + 275$, where h is in feet.
   A. How long will it take the object to hit the ground?
   B. At what times is the height of the object increasing?
   C. At what times is the height of the object decreasing?

8. A ball is thrown across a field. The height of the ball at time t, in seconds, is given by $h(t) = -4.9t^2 + 15t + 2$ where h is in meters.
   A. If someone catches the ball at a height of 3 meters on its way down, how long was the ball in the air?
   B. Find the time it takes the ball to hit the ground if it is not caught.
   C. At what times was the height of the ball increasing?
   D. When does the ball reach its maximum height? What is the maximum height?

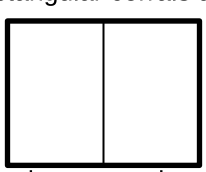
9. The number of bacteria in a refrigerated food is given by $N(t) = 20T^2 - 20T + 120$, for $-2 \le T \le 14$ and where *T* is the temperature of the food in Celsius. At what temperature will the number of bacteria be minimal?

10. An object is dropped off of a 300-foot building and its height at time t, in seconds, is given by $h = -16t^2 + 300$, where h is in feet. How long will it take the object to hit the ground?

11. The height,h , in feet of an object above the ground is given by $h = -16t^2 + 48t + 150$, where *t* is the time in seconds. Find the time it takes the object to strike the ground and find the maximum height of the object.

12. The length of a rectangle is three more than twice the width.
    A. What is the minimum area that this rectangle can have?
    B. Determine the dimensions that will give a total area of 27 m^2.

13. Two rectangular corrals are to be made from 100 yds of fencing as seen below.



If the rancher wants the total area to be maximum, what dimensions should be used to make the corrals?

14. The number of board feet in a 16 foot long tree is approximated by the model $F(d) = 0.77d^2 - 1.32d - 9.31$ where *F* is the number of feet and *d* is the diameter of the log in inches.
    A. How many board feet are in a log with diameter 12 inches?
    B. What is the diameter that will produce the minimum number of board feet?

15. The number of horsepower needed to overcome a wind drag on a certain automobile is given by $H(s) = 0.05s^2 + 0.007s – 0.031$, where *s* is the speed of the car in miles per hour.
    A. How much horsepower is needed to overcome the wind drag on this car if it is traveling 50 miles per hour?
    B. At what speed will the car need to use 200 horsepower to overcome the wind drag?

16. A manufacturer of tennis balls has a daily cost of $C(x) = 200 – 10x + 0.01x^2$ where *C* is the total cost in dollars and *x* is the number of tennis balls produced.
    A. What number of tennis balls will produce the minimum total cost? What is the minimum cost?
    B. On what interval will the total cost be decreasing?

17. A textile manufacturer has daily production costs of $C(x) = 10000 – 110x + 0.05x^2$, where is the total cost (in dollars) and is the number of units produced. How many units should be produced each day to yield a minimum cost?

18. A company earns a weekly profit of $P(x) = -0.75x^2 + 60x – 300$ dollars by selling x items. How many items does the company have to sell each week to maximize the profit?

19. A ball rolls down a slope and travels a distance $d = 6t + ½ t^2$ feet in *t* seconds. Find when the distance is 17 feet.

20. The video screen in the Dallas Cowboys stadium is 11520 square feet. The length is 88 feet more than the width. Find the dimensions of the video screen.

21. The path of a high diver is given by $y = -\frac{4}{9}x^2 + \frac{24}{9}x + 10$ where y is the height in feet above the water and x is the horizontal distance from the end of the diving board in feet. What is the maximum height of the diver and how far out from the end of the diving board is the diver when he hits the water?

22. The length of a rectangular plot of land is 10 yards more than its width. If the area of the land is 600 square yards, find the dimensions of the plot of land.

23. The height of a triangular window is 3 feet less than its base. If the area of the window is 20 square feet, find the dimensions of the window.

24. The length of a Ping-Pong table is 3 ft more than twice the width. The area of a Ping-Pong table is 90 square feet. What are the dimensions of a Ping-Pong table?

25. Three hundred feet of fencing is available to enclose a rectangular yard along side of the St. John’s River, which is one side of the rectangle.
    A. What dimensions will produce an area of $10{,}000\text{ft}^2$?
    B. What is the maximum area that can be enclosed?

23. Five hundred feet of fencing is available to enclose a rectangular lot along side of highway 65. Cal Trans will supply the fencing for the side along the highway, so only three sides are needed.
    A. What dimensions will produce an area of $40{,}000\text{ ft}^2$?
    B. What is the maximum area that can be enclosed?

6.6 Quadratic Inequalities

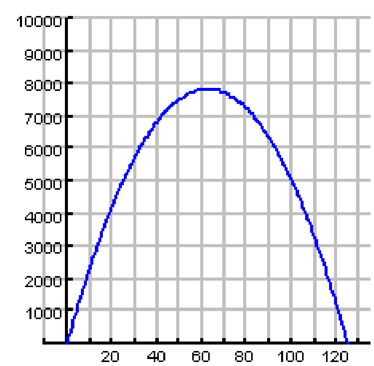
Quadratic inequalities can be solved graphically, numerically, and algebraically. We are going to focus on solving them graphically. Recall that when solving inequalities, the solution is usually an interval of values. The solutions can be written in inequality notation or interval notation.

To solve a quadratic inequality:

1. Graph both sides of the inequality.
2. Find the intersection of the graphs.
3. Determine which intervals on the x-axis solve the inequality.
4. Write the solution using the correct notation.

Examples:

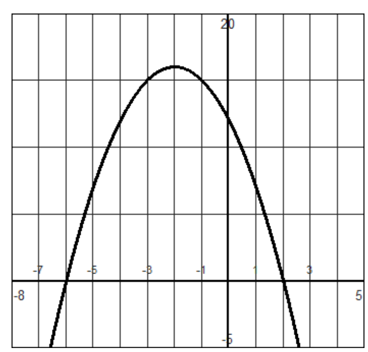
1. A parking lot is rectangular in shape and measures 250 feet around three of the four sides. Assuming the width of the parking lot is x, then the area of the lot can be modeled by $A = (250-2x)x$. A graph of the function is shown to the right. Find the possible widths of the lot if the area must be at least 5000 square feet.



Solution:

First, we find where the area of the parking lot is equal to 5000 square feet. Sketch a horizontal line at 5000 square feet and find the intersection of the two graphs. The intersection points occur where x = 25 feet and where x = 100 feet. The area will be greater than 5000 square feet for x-values in between 25 and 100 feet. Therefore, the possible widths are [25,100].

2. Solve $-x^2 - 4x + 12 < 0$ graphically.



Solution:

First, sketch a graph of $y = -x^2 - 4x + 12$.

Next, find the values of x where the equation is zero (these are the x-intercepts). We can see that the equation is zero at x = -6 and x = 2. Since, we want the intervals where the equation is less than zero, look at each interval which are divided by the x-intercepts. On the interval (-∞, -6) the y-values are less than zero, on the interval (-6, 2) the y-values are greater than zero, and on the interval (2, ∞) the y-values are less than zero. The solution to the inequality is (-∞, -6) and (2, ∞).

Homework:

1. An object is dropped off of a 275-foot building and its height at time t, in seconds, is given by

   $h = -16t^2 + 275$, where h is in feet.

   A. At what times is the object above 200 feet?

   B. At what times is the object below 100 feet?

2. A ball is thrown across a field. The height of the ball at time t, in seconds, is given by $h(t) = -4.9t^2 + 15t + 2$ where h is in meters.

   A. If someone can catch the ball on its way down as long as it is below 2.5 meters, at what times can the ball be caught?

   B. At what times was the ball above 8 meters in the air?

3. The number of bacteria in a refrigerated food is given by $N(t) = 20T^2 – 20T + 120$, for $-2 \le T \le 14$ and where *T* is the temperature of the food in Celsius. If the food is safe to eat as long as it has less than 500 bacteria, at what temperatures should the food be kept?

4. The height,h , in feet of an object above the ground is given by $h = -16t^2 + 68t + 150$, where *t* is the time in seconds. When is the object above 200 feet?

5. The number of board feet in a 16 foot long tree is approximated by the model $F(d) = 0.77d^2 -1.32d – 9.31$ where *F* is the number of feet and *d* is the diameter of the log in inches. If you must have at least 12 board feet, what are possible diameters for the logs?

6. A manufacturer of tennis balls has a daily cost of $C(x) = 200 – 10x + 0.01x^2$ where *C* is the total cost in dollars and *x* is the number of tennis balls produced. For how many tennis balls can cost be kept under $300?

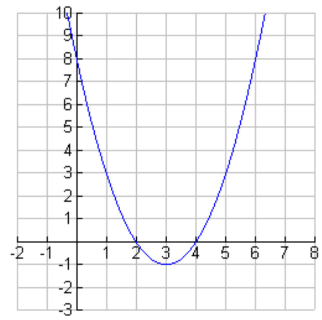
7. A company earns a weekly profit of $P(x) = -0.75x^2 + 65x – 300$ dollars by selling x items. How many items does the company have to sell each week to have at least a $700 profit?

8. The path of a high diver is given by $y = -\frac{4}{9}x^2 + \frac{24}{9}x + 10$ where y is the height in feet above the water and x is the horizontal distance from the end of the diving board in feet. How far from the board (horizontally) is the diver when he is less than 4 feet above the water?

9. Three hundred feet of fencing is available to enclose a rectangular yard along side of the St. John's River, which is one side of the rectangle. What dimensions will produce an area of at most $10,000 ft^2$?

10. Five hundred feet of fencing is available to enclose a rectangular lot along side of highway 65. Cal Trans will supply the fencing for the side along the highway, so only three sides are needed. What dimensions will produce an area of at least 40,000 ft^2?

11. Use the graph to solve $x^2 - 6x + 8 < 3$.



12. Solve $2x^2 – 5x + 2 \leq 0$ graphically.

13. Solve $-3x^2 < 2x + 7$ graphically.